

THE GR-SEGMENTS FOR TAME QUIVERS

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ABSTRACT. A GR-segment for an artin algebra is a sequence of Gabriel-Roiter measures, which is closed under direct predecessors and successors. The number of the GR-segments indexed by natural numbers \mathbb{N} and integers \mathbb{Z} probably relates to the representation types of artin algebras. Let k be an algebraically closed field and Q be a tame quiver (of type $\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$, or $\tilde{\mathbb{E}}_8$). Let b be the number of the isomorphism classes of the exceptional quasi-simple modules over the path algebra $\Lambda = kQ$. We show that the number of the \mathbb{N} - and \mathbb{Z} -indexed GR-segments in the central part for Q is bounded by $b + 1$. Therefore, there are at most $b + 3$ GR segments.

Keywords. tame quiver, Gabriel-Roiter measure, direct predecessor, GR segment.

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1. PRELIMINARIES AND MAIN THEOREM

We first recall what Gabriel-Roiter measures are [7, 8]. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers and $\mathcal{P}(\mathbb{N})$ be the set of all subsets of \mathbb{N} . A total order on $\mathcal{P}(\mathbb{N})$ can be defined as follows: if I, J are two different subsets of \mathbb{N} , write $I < J$ if the smallest element in $(I \setminus J) \cup (J \setminus I)$ belongs to J . Also we write $I \ll J$ provided $I \subset J$ and for all elements $a \in I, b \in J \setminus I$, we have $a < b$. We say that J starts with I if $I = J$ or $I \ll J$.

Let Λ be a connected artin algebra and $\text{mod } \Lambda$ be the category of finite generated left Λ -modules. We denote by $|M|$ the length of a Λ -module M . For each $M \in \text{mod } \Lambda$, let $\mu(M)$ be the maximum of the sets $\{|M_1|, |M_2|, \dots, |M_t|\}$, where $M_1 \subset M_2 \subset \dots \subset M_t$ is a chain of indecomposable submodules of M . We call $\mu(M)$ the *Gabriel-Roiter* (GR for short) *measure* of M . If M is an indecomposable Λ -module, we call an inclusion $T \subset M$ with T indecomposable a *GR inclusion* provided $\mu(M) = \mu(T) \cup \{|M|\}$, thus if and only if every proper submodule of M has Gabriel-Roiter measure at most $\mu(T)$. In this case, we call T a *GR submodule* of M .

An element $I \in \mathcal{P}(\mathbb{N})$ is called a GR measure for Λ if there is an indecomposable Λ -module M with $\mu(M) = I$. Given a GR measure I , we denote by $\mathcal{A}(I)$ the set of representatives of (the isomorphism classes) of the indecomposable modules with GR measure I . We also denote by $|I|$ the maximal element of I , i.e., the length of M with $M \in \mathcal{A}(I)$. The following is a direct consequence of the definitions.

Lemma 1.1. *Let $I < I' < J$ be GR measures for Λ .*

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- (1) If J starts with I , then I' starts with I .
- (2) If $J = I \cup \{|J|\}$, then $|I'| > |J|$.

In [8], the following theorem was proved:

Theorem 1.2. *Let Λ be a representation-infinite artin algebra. Then there are Gabriel-Roiter measures I_i and I^i :*

$$I_1 < I_2 < I_3 < \dots \quad \dots < I^3 < I^2 < I^1$$

such that any other GR measure I satisfies $I_i < I < I^i$ for all i .

The GR measures I_i (resp. I^i) are called *take-off* (resp. *landing*) measures. Any other GR measure is called a *central measure*. An indecomposable module M is called a take-off (resp. central, landing) module if $\mu(M)$ is a take-off (resp. central, landing) measure.

Let I and J be two GR measures for Λ . Then J is called a *direct successor* of I if first, $I < J$ and second, there is no other GR measure I' such that $I < I' < J$. The so-called Successor Lemma in [9] claims that any GR measure different from I^1 , the maximal one, has a direct successor. However, a GR measure, which is not the minimal one I_1 , may not admit a direct predecessor.

A sequence of GR measures for Λ is called a *GR segment* if it is closed under taking direct predecessors and successors. By Theorem 1.2 and the Successor Lemma, a GR-segment \mathcal{S} is finite if and only if it is the only GR segment, and if and only if Λ is of finite representation type.

Fix a representation-infinite artin algebra. Starting with a GR measure μ_0 , we may obtain a sequence of GR measures by taking direct successors and predecessors:

$$\dots < \mu_{-3} < \mu_{-2} < \mu_{-1} < \mu_0 < \mu_1 < \mu_2 < \mu_3 < \dots$$

If μ_0 is not a landing measure, then μ_i exist for all $i \geq 1$ by Successor Lemma. However, μ_{-j} may not exist for some $r \geq 1$ and any $j \geq r$, since there are GR measures admitting no direct predecessors. A infinite GR segment can be naturally said to be indexed by natural numbers \mathbb{N} , $-\mathbb{N}$ or by integers \mathbb{Z} .

From now on, a GR segment always means an infinite one. The following observations are straightforward:

- The unique $-\mathbb{N}$ -indexed GR segment is the landing part.
- The GR-segment containing a take-off measure is \mathbb{N} -indexed.
- The \mathbb{N} -indexed GR segments one-to-one correspond to the GR measures admitting no direct predecessors.
- A GR segment containing a central measure is either \mathbb{N} - or \mathbb{Z} -indexed.

The number of the \mathbb{N} - and \mathbb{Z} -indexed GR segments was thought to relate the representation types of finite dimensional algebras (or more general, artin algebras) [4, 5]. It was

conjectured that a quiver is of wild type if and only if there are infinitely many \mathbb{N} - or \mathbb{Z} -indexed GR segments. It was shown in [4] that for a tame quiver (of type $\tilde{\mathbb{A}}_n, \tilde{\mathbb{D}}_n, \tilde{\mathbb{E}}_6, \tilde{\mathbb{E}}_7$, or $\tilde{\mathbb{E}}_8$) there are, but only finitely many, GR measures admitting no direct predecessors. This precisely means that the number of \mathbb{N} -indexed GR segments is finite. It was also proved in [5] that for wild n -Kronecker quivers there are infinitely many \mathbb{N} -indexed GR segments.

From now on, let k be an algebraically closed field and Q be tame quiver. We refer to [1, 6, 7] for basic concepts of representation theory of (tame) quivers. Let X be a quasi-simple module. We denote by R_X the rank of X , i.e., the minimal natural number such that $\tau^{R_X} X \cong X$, where τ is the Auslander-Reiten translation. Any indecomposable regular module M is of the form X_i , where X is quasi-simple and i is the quasi-length of M , i.e., the length of the unique sequence of irreducible monomorphisms $X = X_1 \rightarrow X_2 \rightarrow \dots \rightarrow X_i = M$. Let $M = X_i$ for some quasi-simple module X . M is called exceptional if $R_X \geq 2$. Otherwise, M is called homogeneous and denote by H_i . If X is quasi-simple, the dimension vector $\underline{\dim} X_{R_X} = \delta$, where δ is the minimal imaginary root of Q . We also denote by $|\delta|$ the sum of all coordinates of δ . Thus it is the length X_{R_X} . Let b be the number of the isomorphism classes of the exceptional quasi-simple modules and a be the number of the isomorphism classes of the exceptional quasi-simple modules X whose GR measures satisfy $\mu(X_{R_X}) \geq \mu(H_1)$. We list the value of b as follows, where p is the number of the clockwise arrows and q is the number of anti-clockwise arrows of type $\tilde{\mathbb{A}}_{p,q}$:

	$\tilde{\mathbb{A}}_n = \tilde{\mathbb{A}}_{p,q}$			$\tilde{\mathbb{D}}_n$	$\tilde{\mathbb{E}}_6$	$\tilde{\mathbb{E}}_7$	$\tilde{\mathbb{E}}_8$
b	$p = q = 1$	$p = 1, q > 1$ or $q = 1, p > 1$	$p, q > 1$	$n + 2$	8	9	10
	0	$p + q - 1$	$p + q$				

In this paper, we will again focus on tame quivers and study the structure of \mathbb{N} - and \mathbb{Z} -indexed GR segments. The following theorem will be proved:

Theorem. *Let Q be a tame quiver. The number of the \mathbb{Z} -indexed GR segments is bounded by a . The number of the \mathbb{N} and \mathbb{Z} -indexed GR segments in the central part is bounded by $b + 1$.*

The direct successors of the GR measures of regular modules will be described in Section 2. Section 3 is devoted to a discussion of the structure of \mathbb{N} - and \mathbb{Z} -indexed GR segments and a proof of the main theorem.

2. DIRECT SUCCESSORS OF GR MEASURES OF REGULAR MODULES

In this section, we study the direct successors of $\mu(X_i)$, where X is a quasi-simple module and i large enough. The results in the section were first shown for quivers of type $\tilde{\mathbb{A}}_n$ in [4] and claimed being true for all tame quivers. We include the proofs for the convenience for later discussion. Throughout this section, we fix a tame quiver Q .

We collect some known facts in the following proposition, which will be quite often used in our later discussion. The proofs can be found in [3].

- Proposition 2.1.** (1) If M is an indecomposable preprojective module, then M is a take-off module and $\mu(M) < \mu(H_1)$.
- (2) Let H_1 be a homogeneous quasi-simple module. Then $\mu(H_1)$ is a central measure and $\mu(H_{i+1})$ is a direct successor of $\mu(H_i)$ for each $i \geq 1$. Moreover, there are only finitely many indecomposable preinjective modules M with $\mu(M) < \mu(H_1)$.
- (3) Let X be quasi-simple and T be a GR submodule of X_i for some $i \geq 1$. Then T is either preprojective or $T \cong X_{i-1}$.
- (4) Let X be a quasi-simple module.
- a) If $\mu(X_r) < \mu(H_1)$, then $\mu(X_i) < \mu(H_j)$ for all $i \geq 1$ and $j \geq 1$.
 - b) If $\mu(X_r) \geq \mu(H_1)$, then X_{i-1} is the unique (up to isomorphism) GR submodule of X_i for every $i \geq r$. If, in addition, $r > 1$, then $\mu(X_i) > \mu(H_j)$ for all $i > r$ and $j \geq 1$.
- (5) Let M be preinjective, which is not in take-off part. If X_i is a GR submodule of M for some quasi-simple module X . Then $\mu(M) > \mu(X_j)$ for all $j \geq 1$.

Lemma 2.2. Let X, Y be quasi-simple modules with rank r and s , respectively. Assume that $\mu(X_r) \geq \mu(H_1)$.

- (1) If $\mu(X_r) > \mu(Y_s)$, then $\mu(X_i) > \mu(Y_j)$ for all $i \geq r, j \geq 1$.
- (2) If $\mu(X_i) = \mu(Y_j)$ for some $i \geq 2r$, then $r = s$ and $\mu(X_t) = \mu(Y_t)$ for every $t \geq r$.
- (3) If $\mu(X_{2r}) > \mu(Y_{2s})$, then $\mu(X_i) > \mu(Y_j)$ for all $i \geq 2r, j \geq 1$.

Proof. (1) If $\mu(Y_s) < \mu(H_1)$, then $\mu(Y_j) < \mu(H_1)$ for all $j \geq 1$. Thus we may assume that $\mu(Y_s) \geq \mu(H_1)$. Since for each $j \geq s$, $\mu(Y_j)$ starts with $\mu(Y_s)$ and $|Y_s| = |X_r| = |\delta|$, we have $\mu(X_r) > \mu(Y_j)$.

(2) It is clear that $r = 1$ if and only if $s = 1$. Now we assume $r > 1$. Since $\mu(X_r) \geq \mu(H_1)$, we have $\mu(Y_s) \geq \mu(H_1)$. Thus $j \geq 2s$ and

$$\begin{aligned} \mu(Y_j) &= \mu(Y_s) \cup \{|Y_{s+1}|, |Y_{s+2}|, \dots, |Y_{2s}|, |Y_{2s+1}|, \dots, |Y_j|\} \\ &= \mu(X_r) \cup \{|X_{r+1}|, |X_{r+2}|, \dots, |X_{2r}|, |X_{2r+1}|, \dots, |X_i|\} = \mu(X_i). \end{aligned}$$

Because $|X_r| = |Y_s| = |\delta|$ and $|X_{2r}| = |Y_{2s}| = 2|\delta|$, we obtain that $r = s$, $\mu(X_r) = \mu(Y_s)$ and $\mu(X_{2r}) = \mu(Y_{2s})$. Note that

$$|X_{r+l}| - |X_{r+l-1}| = |Y_{r+l}| - |Y_{r+l-1}|$$

for all $l \geq 1$. It follows $\mu(X_t) = \mu(Y_t)$ for all $t \geq r = s$.

(3) follows similarly. □

Corollary 2.3. Let X be a quasi-simple module of rank r such that $\mu(X_r) \geq \mu(H_1)$. If M is an indecomposable module such that $\mu(M) = \mu(X_i)$ for some $i \geq 2r$, then M is a regular module.

Proof. For the purpose of a contradiction, let M be an indecomposable preinjective module with $|M|$ minimal such that $\mu(M) = \mu(X_i)$ for some $i \geq 2r$. Note that $i - 1 \geq 2r$, since

$|M| \neq 2|\delta|$. Let T be a GR submodule of M . Then $\mu(T) = \mu(X_{i-1}) > \mu(H_1)$. By the minimality of $|M|$, T is regular, say $T = Y_t$ for some quasi-simple module Y of rank s . Then $\mu(M) > \mu(Y_j)$ for all $j \geq 1$ by Proposition 2.1(5). Thus $Y \not\cong X$ and $t \geq 2s$ since $|M| = |X_i| > 2|\delta|$. It follows that $\mu(Y_s) \geq \mu(H_1)$. Notice that $\mu(Y_t) = \mu(X_{i-1})$. Therefore, $r = s$ and $\mu(Y_{t+1}) = \mu(X_i)$ by Lemma 2.2 which implies $|Y_{t+1}| = |X_i| = |M|$. On the other hand, it is easily seen that $|Y_{t+1}| > |M|$. This is a contradiction. \square

Proposition 2.4. *Let X be a quasi-simple module of rank $r > 1$.*

- (1) *If $\mu(X_r) \geq \mu(H_1)$. Then $\mu(X_{j+1})$ is a direct successor of $\mu(X_j)$ for each $j \geq 2r$.*
- (2) *If $\mu(X_r) < \mu(H_1)$ and if there is an $i \geq 1$ such that X_i is a central module. Then there is an $i_0 \geq i$ such that $\mu(X_{j+1})$ is a direct successor of $\mu(X_j)$ for each $j \geq i_0$.*

Proof. (1) We first show that there does not exist an indecomposable regular module M such that $\mu(M)$ lies between $\mu(X_j)$ and $\mu(X_{j+1})$ for any $j \geq 2r$. For the purpose of a contradiction, we assume that there exists a $j \geq 2r$ and an indecomposable regular module M with $|M|$ minimal and $\mu(X_j) < \mu(M) < \mu(X_{j+1})$. Then $|M| > |X_{j+1}| > 2|\delta|$, since X_j is a GR submodule of X_{j+1} . Let $M = Y_t$ for some quasi-simple module Y of rank $s > 1$. It follows that $\mu(Y_s) \geq \mu(H_1)$ and $t > 2s$. Therefore, Y_{t-1} is a GR submodule of Y_i and

$$\mu(Y_{t-1}) \leq \mu(X_j) < \mu(M) = \mu(Y_t) < \mu(X_{j+1})$$

by minimality of $|M|$. This implies $\mu(Y_{t-1}) = \mu(X_j)$, since otherwise $|X_j| > |M| > |X_{j+1}|$, which is impossible. Observe that $t - 1 \geq 2s$ and $j \geq 2r$. Then Lemma 2.2 implies $\mu(X_i) = \mu(Y_i)$ for all $i \geq r = s$. This contradicts the assumption $\mu(X_j) < \mu(M) = \mu(Y_t) < \mu(X_{j+1})$. Therefore, there are no indecomposable regular modules M satisfying $\mu(X_j) < \mu(M) < \mu(X_{j+1})$ for any $j \geq 2r$.

Assume that M is an indecomposable preinjective module such that $\mu(X_j) < \mu(M) < \mu(X_{j+1})$ with $|M|$ minimal. Let N be a GR submodule of M . Comparing the lengths, we have $\mu(X_j) \leq \mu(N)$. If $N = Y_h$ is regular for some quasi-simple module Y of rank s , then $\mu(X_{j+1}) > \mu(M) > \mu(Y_{h+1}) > \mu(Y_h) \geq \mu(X_j)$. This contradicts the first part of the proof. If N is preinjective, then $\mu(N) = \mu(X_j)$ by the minimality of $|M|$. Thus a GR filtration of N contains a regular module Z_{2t} for a quasi-simple module Z of rank t . It follows that $\mu(X_{2r}) = \mu(Z_{2t})$. Thus $\mu(M) > \mu(N) > \mu(Z_{i+1}) = \mu(X_{i+1})$, which is a contradiction.

(2) Since there are only finitely many indecomposable preinjective modules with GR measures smaller than $\mu(H_1)$, we may choose $j_0 \geq i$ such that $\mu(X_j) < \mu(M) < \mu(X_{j+1})$ implies that M is regular for any $j \geq j_0$. It is sufficient to show that there is an $i_0 \geq j_0$ such that there does not exist a regular module with GR measure μ satisfying $\mu(X_j) < \mu < \mu(X_{j+1})$ for any $j \geq i_0$.

Since X_i is a central module, X_j is the unique, up to isomorphism, GR submodule of X_{j+1} for every $j \geq j_0$. Let Y be a quasi-simple module of rank s such that $\mu(X_j) < \mu(Y_l) < \mu(X_{j+1})$ for some $j \geq j_0 \geq r$ and $l \geq 1$. In this case, Y_l is a GR submodule of

Y_{l+1} since Y_l is a central module. Comparing the lengths, we have $\mu(Y_{l+1}) < \mu(X_{j+1})$, and similarly $\mu(Y_h) < \mu(X_{j+1})$ for all $h \geq 1$. Now replace j by some $j' > j$ and repeat the above consideration. Since there are only finitely many quasi-simple modules Z such that $\mu(Z_{R_Z}) \leq \mu(H_1)$, where R_Z is the rank of Z , we may obtain an index $i_0 > j_0$ such that a GR measure μ of an indecomposable regular module satisfies either $\mu < \mu(X_{i_0})$ or $\mu > \mu(X_j)$ for all $j \geq 1$. \square

3. THE STRUCTURES OF GR-SEGMENTS

In this section, we study the structure of the \mathbb{N} - and \mathbb{Z} -indexed GR segments for a fixed tame quiver Q . The main theorem will be also proved in this section.

3.1. Sequence of direct successors. Let μ_0 be a central measure and \mathcal{S} be the sequence of GR measures obtained by taking direct successors starting with μ_0 :

$$\mu_0 < \mu_1 < \mu_2 < \mu_3 < \mu_4 \dots$$

Lemma 3.1. *For each $\mu \in \mathcal{S}$, $\mu > I_i$ for all take-off measures I_i . In particular, M is not a preprojective module for any $M \in \mathcal{A}(\mu)$.*

Proof. This is straightforward since μ is not a take-off measure and all indecomposable preprojective modules are take-off modules (Proposition 2.1). \square

Lemma 3.2. *Let X be a quasi-simple module with $\mu(X_s) = \mu_i \in \mathcal{S}$. Assume that N is an indecomposable regular module with $\mu(N) = \mu_j$ for some $j < i$ such that $\mathcal{A}(\mu_h)$ contains no regular modules for any $j < h < i$. Then $\mu(N) = \mu(X_{s-1})$. In particular, if $s = 1$, then $\mathcal{A}(\mu_j)$ contains no regular modules for any $j < i$.*

Proof. Assume that $N = Y_t$ for some quasi-simple module Y and $t \geq 1$. Let $T \subset X_s$ be a GR submodule. Then T is either preprojective, or isomorphic to X_{s-1} . By the choice of i and j and the fact that \mathcal{S} contains no take-off measure, we have $\mu(T) \leq \mu(Y_t) < \mu(X_s)$. If the equality does not hold (for example, $s = 1$ and thus T is preprojective), then $|Y_t| > |X_s|$ by Lemma 1.1 since T is a GR submodule of X_s . It follows from the assumption that $\mu(Y_t) < \mu(X_s) < \mu(Y_{t+1})$. Again consider a GR submodule of Y_{t+1} . Similar to the above situation, we have $|X_s| > |Y_{t+1}|$, which contradicts $|Y_t| > |X_s|$. Thus $\mu(T) = \mu(Y_t)$ and $T \cong X_{s-1}$ since Y_t is a central module. \square

As a direct consequence of this lemma, we can show the existence of an \mathbb{N} -indexed GR segment which is not the take-off part.

Corollary 3.3. *A GR segment containing $\mu(H_i)$ is indexed by \mathbb{N} .*

Proof. It is known that $\mu(H_{i+1})$ is a direct successor of $\mu(H_i)$ for all $i \geq 1$. Thus a GR segment $\mathcal{S}_{\mathbb{Z}}$ contains $\mu(H_i)$ for some i if and only if it contains all $\mu(H_i)$. Without loss of generality, we may assume $\mu(H_1) = \mu_0 \in \mathcal{S}_{\mathbb{Z}}$. By Lemma 3.2, for each GR measure μ_{-j}

obtained by taking direct predecessors from μ_0 contains only preinjective modules. Thus there are infinitely many indecomposable preinjective modules with GR measures smaller than $\mu(H_1)$. This is a contradiction. \square

Remark. For a tame quiver of type \tilde{A}_n , $\mu(H_1)$ does always not admit a direct predecessors [4].

Lemma 3.4. *Assume that $|\mu_0| \leq |\mu_i|$ for all $i \geq 0$. Then for each $i \geq 1$, μ_i starts with μ_0 .*

Proof. We use induction on i . Since $\mu_1 \neq I_1$, we may write $\mu_1 = \mu'_1 \cup \{|\mu_1|\}$. Thus $\mu'_1 \leq \mu_0 < \mu_1$. If the equality does not hold, then $|\mu_0| > |\mu_1|$. This contradicts the minimality of $|\mu_0|$. Thus $\mu'_1 = \mu_0$ and μ_1 starts with μ_0 . Now assume that $i > 1$ and μ_r starts with μ_0 for all $1 \leq r \leq i$. Let $\mu_{i+1} = \mu'_{i+1} \cup \{|\mu_{i+1}|\}$. If $\mu'_{i+1} \leq \mu_0 < \mu_{i+1}$, then we are done. Otherwise, $\mu'_{i+1} = \mu_r$ for some $1 \leq r \leq i$. Hence, μ'_{i+1} and thus μ_{i+1} starts with μ_0 by induction. \square

Lemma 3.5. *For each i , there is some $j > i$ such that $\mathcal{A}(\mu_j)$ contains regular modules.*

Proof. To obtain a contradiction, we may assume, without loss of generality, that $|\mu_0| \leq |\mu_i|$ and that $\mathcal{A}(\mu_i)$ contains only preinjective modules for all $i \geq 0$. By previous lemma, μ_i starts with μ_0 for all $i \geq 1$. Since $\mathcal{A}(\mu_0)$ contains only finitely many indecomposable preinjective modules, there are infinitely many indecomposable preinjective modules containing a preinjective module $M \in \mathcal{A}(\mu_0)$ as a submodule. This is impossible. \square

Namely, we may show a much stronger consequence.

Lemma 3.6. *There is some i such that $\mathcal{A}(\mu_j)$ contains only regular modules for all $j \geq i$.*

Proof. Since for any indecomposable preinjective module N , $\mu(N) \neq \mu(H_j)$ for any j , we may assume that $\mu_i \neq \mu(H_j)$ for any i, j . Thus either $\mu_0 < \mu(H_1)$ or $\mu_0 > \mu(H_1)$.

Assume that $\mu_0 > \mu(H_1)$. By Lemma 3.5, we may assume that $\mathcal{A}(\mu_i)$ contains a regular module M such that $|M| > 2|\delta|$ for some i . We may write $M = X_s$ for some quasi-simple module X and $s > 2R_X$. On the other hand, $\mu_j > \mu(H_1)$ for all j . Therefore, $\mu(X_{j+1})$ is a direct successor of $\mu(X_j)$ for all $j \geq 2R_X$ (Proposition 2.4). It follows that $\mu(X_{s+j}) = \mu_{r+j}$. Note that there does not exist an indecomposable preinjective module M with GR measure $\mu(M) = \mu(X_{s+j})$ for any $j \geq 0$ (Corollary 2.3).

If $\mu_0 < \mu(H_1)$, then $\mu_j < \mu(H_1)$ for all j . Since there are only finitely many indecomposable preinjective module N with $\mu(N) < \mu(H_1)$ (Proposition 2.1), we may obtain some i such that $\mathcal{A}(\mu_j)$ consists of regular modules for each $j \geq i$.

The proof is completed. \square

3.2. \mathbb{Z} -indexed GR segments. Let $\mathcal{S}_{\mathbb{Z}}$ be a \mathbb{Z} -indexed GR segment:

$$\dots < \mu_{-3} < \mu_{-2} < \mu_{-1} < \mu_0 < \mu_1 < \mu_2 < \mu_3 < \dots$$

We describes $\mathcal{A}(\mu_i)$ for i smaller enough.

Lemma 3.7. *There is some r such that $\mathcal{A}(\mu_i)$ contains only preinjective modules for all $i < r$.*

Proof. Let X be an indecomposable regular module such that $\mu(X) \in \mathcal{S}_{\mathbb{Z}}$ and such that $|X|$ is minimal. Without loss of generality we may assume that $\mu(X) = \mathcal{A}(\mu_0)$. If there is an indecomposable regular module Y with $\mu(Y) < \mu(X) = \mu_0$ and $\mu(Y) \in \mathcal{S}$, then $|Y| < |X|$ by Lemma 3.2. This is a contradiction. Thus $\mathcal{A}(\mu_{-i})$ contains no regular modules for any $i > 0$. \square

Corollary 3.8. *For all i , $\mu_i > \mu(H_1)$.*

Proof. Since there are only finitely many indecomposable preinjective modules with GR measures smaller than $\mu(H_1)$, we have $\mu_{-i} > \mu(H_1)$ for $i > 0$ larger enough by previous lemma. Thus $\mu_i > \mu(H_1)$ for all i . \square

Let a be the number of the isomorphism classes of the exceptional quasi-simple modules X whose GR measures satisfy $\mu(X_{R_X}) \geq \mu(H_1)$.

Proposition 3.9. *The number of the \mathbb{Z} -indexed GR segments is bounded by a .*

Proof. Assume that $\mathcal{S}_{\mathbb{Z}}$ is a \mathbb{Z} -indexed GR segment. Since $\mu_i > \mu(H_1)$, there is a $\mu_i \in \mathcal{S}_{\mathbb{Z}}$ such that $\mathcal{A}(\mu_i)$ contains a regular module X_s and $\mu_{i+j} = \mu(X_{s+j})$ for some quasi-simple module X . Therefore, $\mathcal{S}_{\mathbb{Z}}$ gives (not unique in general) an exceptional quasi-simple X such that $\mu(X_{R_X}) > \mu(H_1)$. It is clear that different \mathbb{Z} -indexed GR segments correspond to non-isomorphic quasi-simple modules. Therefore, there are at most a \mathbb{Z} -indexed GR segments. \square

3.3. N-indexed GR segments. Corollary 3.3 shows the existence of an \mathbb{N} -indexed GR segment. It was already proved in [4] that for a tame quiver there are, but only finitely many, \mathbb{N} -indexed GR segments. However, an upper bound of the number of this kind of GR segments is still missing. Similar to the discussion for \mathbb{Z} -indexed GR segments, we will describe the \mathbb{N} -indexed GR segments containing central measures and give an upper bound of the number. Let $\mathcal{S}_{\mathbb{N}}$: $\mu_0 < \mu_1 < \mu_2 < \dots$ be an \mathbb{N} -indexed GR segment (μ_0 has no direct predecessor), which contains no take-off measures.

Lemma 3.10. *Only finitely many $\mathcal{A}(\mu_i)$ contains preinjective modules.*

Proof. This is just a restatement of Lemma 3.6. \square

Proposition 3.11. *There is some $i > 0$ and some quasi-simple module X such that $\mu_{i+j} = \mu(X_{t+j})$ for some $t \geq 1$ and all $j \geq 0$.*

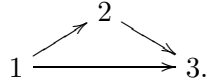
Proof. We may assume without loss of generality that $\mathcal{A}(\mu_i)$ only contain regular modules for all i . By Proposition 2.4, we may also select $s > 0$ such that, for any quasi-simple module X , $\mu(X_{j+1})$ is a direct successor of $\mu(X_j)$ for any $j \geq s$. Let $i > 0$ such that $|\mu_j| > |X_s|$

for all $j \geq i$ and all exceptional quasi-simple modules X . Assume that X is a quasi-simple module such that $X_t \in \mathcal{A}(\mu_i)$, then $X_{t+j} \in \mathcal{A}(\mu_{i+j})$ for all $j \geq 0$. In particular $\mathcal{S}_{\mathbb{N}}$ gives rise to a quasi-simple module X . \square

Proof of Theorem. A \mathbb{Z} -indexed GR segment or an \mathbb{N} -indexed GR segment that contains central measures, which are not of the forms $\mu(H_i)$, gives rise to (may not be unique) an exceptional quasi-simple module. Moreover, different such GR segments correspond to non-isomorphic quasi-simple modules. Thus the number of these kinds of GR segments is bounded by b , the number of the isomorphism classes of the exceptional quasi-simple modules. On the other hand, all $\mu(H_i)$ are contained in the same \mathbb{N} -indexed GR segment. Thus the central part of a tame quiver contains at most $b + 1$ GR-segments. Note that the take-off part is also \mathbb{N} -indexed and the landing part is $-\mathbb{N}$ -indexed. Therefore, a tame quiver has at most $b + 3$ GR-segments.

3.4. Examples. (1) Let Q be a tame quiver of type $\tilde{\mathbb{A}}_n$ with sink-source orientation. If $n = 1$, i.e., Q is a Kronecker quiver, then there is precisely one \mathbb{N} -indexed GR segment, which consists of the GR measures $\mu(H_i)$ of homogeneous modules. If $n > 1$, then the central part contains only two \mathbb{N} -indexed GR segments: one is, as above, consisting of GR measures of homogeneous $\mu(H_i)$, and the other one is of the form $\{\mu(X_i)\}$, where X is any exceptional quasi-simple module. Note that in this case, there are no \mathbb{Z} -indexed GR segments.

(2) Let Q be the following quiver:



Let X be, up to isomorphism, the unique quasi-simple of length 2. We denote by M^i the unique (up to isomorphism) indecomposable preinjective module with length $3i + 2$. The only \mathbb{Z} -indexed GR segment is the following:

$$\dots < \mu(M^i) < \dots < \mu(M^2) < \mu(M^1) = \mu(X_3) < \mu(X_4) < \mu(X_5) < \dots < \mu(X_j) < \dots$$

In the central part, there is precisely one \mathbb{N} -indexed GR segment which is given by the GR measures of homogeneous modules:

$$\mu(X_2) = \mu(H_1) < \mu(H_2) < \dots < \mu(H_i) < \dots$$

We refer to [8] for details of the description of the GR measures of this quiver.

We may characterize the tame quivers of type $\tilde{\mathbb{A}}_n$, which admit \mathbb{Z} -indexed GR segments.

Proposition 3.12. *Let Q be a tame quiver of type $\tilde{\mathbb{A}}_n$. Then the following are equivalent:*

- (1) *Q is not equipped with a sink-source orientation.*
- (2) *There are preinjective central modules.*

- (3) *There are infinitely many isomorphism classes of preinjective central modules.*
- (4) *There exists a \mathbb{Z} -indexed GR segment.*

Proof. The equivalences of the first three statements were already shown in [4].

(4) implies (3) is obvious by Lemma 3.7. Conversely, assume that statement (3) holds. Let $\mathcal{A} = \cup_{\mu} \mathcal{A}(\mu)$, where the union is taken over all central GR measures $\mu \in \mathcal{S}_{\mathbb{N}}$ for some \mathbb{N} -indexed GR segments $\mathcal{S}_{\mathbb{N}}$. This is a finite union since the main theorem gives an upper bound of the number of the \mathbb{N} -indexed GR segments. On the other hand, Lemma 3.10 implies that in each \mathbb{N} -indexed GR segment, there are only finitely GR measures μ such that $\mathcal{A}(\mu)$ contains (finitely many) preinjective modules. It follows that \mathcal{A} contains only finitely many preinjective modules. Therefore, there exists a \mathbb{Z} -indexed GR segment by (3) and the fact that a GR segment in the central part is either \mathbb{N} - or \mathbb{Z} -indexed. \square

REFERENCES

- [1] M.Auslander; I.Reiten; S.O.Smalø, *Representation Theory of Artin Algebras*. Cambridge studies in advanced mathematics **36** (Cambridge University Press, Cambridge, 1995).
- [2] B.Chen, The Gabriel-Roiter measure for representation-finite hereditary algebras. *J. Algebra* **309**(2007), 292-317.
- [3] B.Chen, Comparison of Auslander-Reiten theory and Gabriel-Roiter measure approach to the categories of tame hereditary algebras. *Comm. Algebra* **36**(2008),4186-4200.
- [4] B.Chen, The Gabriel-Roiter measure for $\tilde{\mathbb{A}}_n$ II. *Submitted*.
- [5] B.Chen, The Gabriel-Roiter measures admitting no direct predecessors over n -Kronecker quivers. *Preprint*.
- [6] V.Dlab; C.M.Ringel, *Indecomposable representations of graphs and algebras*. Mem. Amer. Math. Soc. **6**(1976), no.173.
- [7] C.M.Ringel, *Tame algebras and integral quadratic forms*. Lecture Notes in Mathematics, **1099**. Springer-Verlag, Berlin, 1984.
- [8] C.M.Ringel, The Gabriel-Roiter measure. *Bull. Sci. math.* **129**(2005), 726-748.
- [9] C.M.Ringel, Foundation of the representation theory of artin algebras, Using the Gabriel-Roiter measure. Proceedings of the 36th Symposium on Ring Theory and Representation Theory. **2**, 1-19, Symp. Ring Theory Represent Theory Organ. Comm., Yamanashi, 2004.

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